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Real-time Chern-Simons term for hypermagnetic fields

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Abstract: If non-vanishing chemical potentials are assigned to chiral fermions, then a Chern-Simons term is induced for the corresponding gauge fields. In thermal equilibrium anomalous processes adjust the chemical potentials such that the coefficient of the Chern-Simons term vanishes, but it has been argued that there are non-equilibrium epochs in cosmology where this is not the case and that, consequently, certain fermionic number densities and large-scale (hypermagnetic) field strengths get coupled to each other. We generalise the Chern-Simons term to a real-time situation relevant for dynamical considerations, by deriving the anomalous Hard Thermal Loop effective action for the hypermagnetic fields, write down the corresponding equations of motion, and discuss some exponentially growing solutions thereof.

Keywords: Chern-Simons Theories, Global Symmetries, Thermal Field Theory, Baryogenesis.

Contents

1. Introduction

It was realised long ago that if chiral fermions are assigned a non-zero chemical potential, then a Chern-Simons term appears to be induced for the corresponding gauge fields [1]. This statement is not without ambiguities, however [2]. In fact a non-abelian Chern-Simons term transforms non-trivially under large gauge transformations, and therefore the induced theory is well-defined only for certain imaginary values of the chemical potential [3]. Formally, the reason for these problems is that the chiral charge is not conserved because of the axial anomaly [4], so that strictly speaking no chemical potential should be assigned to it.

In the standard electroweak theory, chiral fermions couple not only to non-abelian gauge fields, but also to the abelian hypercharge fields. Hence, at the high temperatures where the electroweak symmetry is restored, a Chern-Simons term appears to be induced for them as well. An abelian Chern-Simons action does not have the same topological properties as the non-abelian one, but on the other hand it is gauge-invariant, and could thus conceivably have a more direct physical significance than its non-abelian counterpart.

Indeed, there have been a number of suggestions for possible roles that the abelian hypermagnetic Chern-Simons term might play in cosmology. One of them is related to the observation that right-handed electrons, which do not take part in weak interactions and also have a very small Yukawa coupling, are practically decoupled from the thermal ensemble above temperatures of about 10 TeV [5]. If they come with a non-vanishing net density, which can be described by a chemical potential, then a hypermagnetic Chern-Simons term gets induced. It has been argued that this leads to an instability and to the subsequent generation of large-scale hypermagnetic fields [6]. While it is believed that any length scales related to physics within the horizon of this epoch are too small to act as seeds for the currently observed galactic magnetic fields [7], such fields could have other physical consequences, for instance affecting the properties of the electroweak phase transition [8], the sphaleron energy [9], and electroweak baryogenesis [10] (in suitable extensions of the Standard Model).

Another possible role acts in the opposite direction. Suppose that there exist primordial hypermagnetic fields as a result for instance of some inflationary dynamics. Then anomalous processes could convert some of these fields to lepton and eventually to baryon number, resulting possibly in the existence of matter and antimatter domains [11], which could lead to nucleosynthesis taking place in a corresponding environment [12].

Whichever of these physics effects is realised, a central ingredient is always the presence of a hypermagnetic Chern-Simons term induced by a fermionic chemical potential. As argued in refs. [6, 11], the Chern-Simons term leads to an additional "anomalous" term in the magnetohydrodynamic equations that govern the evolution of the hypermagnetic fields. The goal of this paper is to attempt a field theoretic derivation of the equations of motion for the hypermagnetic fields in the presence of fermionic chemical potentials. The appropriate framework is that of the Hard Thermal Loop (HTL) effective theories [13, 14]. By integrating out the "hard" fermions, with energies of the order of the temperature T, we derive the effective action for the "soft" gauge fields, with wave vectors $\mathbf{p}$ and minkowskian frequencies ω much smaller than the temperature, $|\mathbf{p}|, |\omega| \ll 2\pi T$. The standard static Chern-Simons term used for instance in the considerations of refs. [6, 11] is recovered if we make the further approximation $|\omega| \ll |\mathbf{p}|$ (which indeed appears to be well justified in practice).

The plan of this note is the following. In section 2 we recapitulate the static gauge field effective action at high temperatures, and in section 3 generalise it to the non-static situation. We solve the resulting equations of motion for a simple case in section 4, and conclude in section 5.

2. Anomalous effective action in the static limit

Let us consider the standard electroweak theory at temperatures T above a few hundred GeV. Let A^a_μ, B_μ be the $\mathrm{SU(2)_L}$ and $\mathrm{U(1)_Y}$ gauge fields, $G^a_{\mu\nu}, F_{\mu\nu}$ the corresponding field strength tensors, and g, g' the gauge couplings. The covariant derivative reads $D_\mu = \partial_\mu - igT^a A^a_\mu + ig'\frac{Y}{2}B_\mu$, where T^a are Hermitean generators normalised as $\mathrm{Tr}\,[T^aT^b] = \delta^{ab}/2$, and Y is the hypercharge quantum number. With these conventions, the gauge field part of the dimensionally reduced euclidean lagrangian [15] takes the form

$$\begin{aligned}\mathcal{L}_E = {} & f_E + \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}m_E^2 A^a_0 A^a_0 + \frac{1}{2}m_E'^2 B_0 B_0 + ij'_E B_0 + \\ & + c_E\, n_{\mathrm{CS}} + c'_E\, n'_{\mathrm{CS}} + \cdots\,, \end{aligned} \tag{2.1}$$

where the "anomalous" Chern-Simons densities read

$$n_{\mathrm{CS}} \equiv \frac{g^2}{32\pi^2}\epsilon_{ijk}\Big(A^a_i G^a_{jk} - \frac{g}{3}f^{abc}A^a_i A^b_j A^c_k\Big)\,, \tag{2.2}$$

$$n'_{\mathrm{CS}} \equiv \frac{g'^2}{32\pi^2}\epsilon_{ijk}\, B_i F_{jk}\,, \tag{2.3}$$

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and infinitely many higher dimensional operators have been suppressed. The term f_E is a (field-independent) "unit operator". The relative parametric accuracy that can be reached with dimensionally reduced effective theories of this kind has been analysed in ref. [16].

There are a number of matching coefficients appearing in eq. (2.1). Denoting by n_G the number of generations, by n_S the number of fundamental scalar doublets, by μ_Q the common chemical potential of the quarks, and by μ_{L_i} (μ_{R_i}) the chemical potential of the $i^{\rm th}$ left-handed (right-handed) lepton generation, the leading-order expressions read

$$\begin{aligned}
f_E &= -\biggl(24+\frac{105}{4}n_\mathrm{G}+4n_\mathrm{S}\biggr)\frac{\pi^2T^4}{90} - \biggl(n_\mathrm{G}\mu_Q^2+\sum_{i=1}^{n_\mathrm{G}}\frac{2\mu_{L_i}^2+\mu_{R_i}^2}{12}\biggr)T^2 - \\
&\quad - n_\mathrm{G}\frac{\mu_Q^4}{2\pi^2}-\sum_{i=1}^{n_\mathrm{G}}\frac{2\mu_{L_i}^4+\mu_{R_i}^4}{24\pi^2}\,, && (2.4)\\
m_E^2 &= g^2\biggl[\biggl(\frac{2}{3}+\frac{n_\mathrm{G}}{3}+\frac{n_\mathrm{S}}{6}\biggr)T^2+3n_\mathrm{G}\frac{\mu_Q^2}{4\pi^2}+\sum_{i=1}^{n_\mathrm{G}}\frac{\mu_{L_i}^2}{4\pi^2}\biggr]\,, && (2.5)\\
m_E'^2 &= g'^2\biggl[\biggl(\frac{5n_\mathrm{G}}{9}+\frac{n_\mathrm{S}}{6}\biggr)T^2+11n_\mathrm{G}\frac{\mu_Q^2}{12\pi^2}+\sum_{i=1}^{n_\mathrm{G}}\frac{\mu_{L_i}^2+2\mu_{R_i}^2}{4\pi^2}\biggr]\,, && (2.6)\\
j_E' &= g'\biggl[\frac{n_\mathrm{G}\mu_Q}{3}\biggl(T^2+\frac{\mu_Q^2}{\pi^2}\biggr)-\sum_{i=1}^{n_\mathrm{G}}\biggl(\frac{\mu_{L_i}+\mu_{R_i}}{6}T^2+\frac{\mu_{L_i}^3+\mu_{R_i}^3}{6\pi^2}\biggr)\biggr]\,, && (2.7)\\
c_E &= 3n_\mathrm{G}\mu_Q+\sum_{i=1}^{n_\mathrm{G}}\mu_{L_i}\,, && (2.8)\\
c_E' &= -c_E+2\sum_{i=1}^{n_\mathrm{G}}\Bigl(\mu_{L_i}-\mu_{R_i}\Bigr)\,. && (2.9)
\end{aligned}$$

Some higher-order corrections can be found in ref. [17].

Now, because of the axial anomaly, the rate of baryon plus lepton number violation, $\mathrm{d}\ln|B+L|/\mathrm{d}t\approx-(13n_\mathrm{G}/4)(25.4\pm2.0)\alpha_w^5T$ [18]–[22],[1] is significantly larger than the expansion rate of the Universe, $\sim T^2/m_\mathrm{Pl}$, for $10^2\,\mathrm{GeV}\lesssim T\lesssim10^{12}\,\mathrm{GeV}$, so that the anomalous processes are perfectly in thermal equilibrium [23]. Therefore, the corresponding chemical potential should be set to zero:

$$\mu_{B+L}\equiv3n_\mathrm{G}\mu_Q+\sum_{i=1}^{n_\mathrm{G}}\mu_{L_i}=0\,. \qquad (2.10)$$

In other words, the coefficient c_E in eq. (2.8) vanishes.

On the other hand, the coefficient c_E' in eq. (2.9) does not vanish, provided that $\mu_{R_i}\neq\mu_{L_i}$. Such a situation can arise if chirality flipping processes, mediated by the Yukawa couplings, are out of equilibrium [5], and we will assume this to be the case in the following. Formally, this can be reached by setting the electron Yukawa coupling to zero.

The naive conversion of eq. (2.1) to Minkowski spacetime goes simply through the analytic continuation

$$\partial_0^E=-i\partial_0^M\,,\qquad A_0^{aE}=-iA_0^{aM}\,,\qquad B_0^E=-iB_0^M\,,\qquad \mathcal{L}_E=-\mathcal{L}_M\,, \qquad (2.11)$$

[1]The number 25.4 is in fact the value of a function containing terms like $\ln(1/\alpha_w)$, at the physical α_w.

and the minkowskian action is then given by $S_M = \int \mathrm{d}t\, \mathrm{d}^3x\, \mathcal{L}_M$. The resulting theory is gauge invariant only in static gauge transformations, however, and thus cannot be the full truth. In the next Section, we recall how a more precise theory can be obtained.

3. Anomalous hard thermal loop effective action

Suppose that we stay for a further moment in the static limit, and consider what kind of higher order operators could appear in eq. (2.1). From the point of view of the original four-dimensional theory, some of these operators arise from a gradient expansion in spatial derivatives. Given that the scale that has been integrated out to obtain eq. (2.1) is the "hard" scale $\sim 2\pi T$, these operators are necessarily suppressed by (at least) $\mathcal{O}(|\nabla|^2/(2\pi T)^2)$ with respect to the ones that have been kept in eq. (2.1).

Now, one might expect that the same is true for temporal derivatives: maybe their effects are also suppressed by $\partial_0^2/(2\pi T)^2$? This is not the case! As is well-known from Hard Thermal Loop considerations [24, 13, 14], time-dependence is suppressed with respect to the static limit only by $\mathcal{O}(|\partial_0|/|\nabla|)$, and is in general of order unity.

Let us proceed with the explicit computation. We employ the Matsubara formalism, followed by analytic continuation. The chemical potential corresponds in momentum space to shifting fermionic Matsubara frequencies ω_n as $\omega_n \to \omega_n + i\mu$. We use the (sometimes implicit) notation that summing over Lorentz indices which are both down implies the use of Euclidean metric. Capital momenta (P, Q, R) are assumed Euclidean.

With these conventions, for any given left-handed fermion with hypercharge Y, the anomalous part of the hypermagnetic Euclidean action in momentum space reads

$$\delta S_E = \frac{1}{4} g'^2 Y^2 \, \SumInt_{Q,R} \delta_{Q+R}\, B_\mu(R) B_\nu(Q) \Gamma_{\mu\nu}(Q)\,, \tag{3.1}$$

$$\Gamma_{\mu\nu}(Q) \equiv \SumInt_{P=(\omega_n+i\mu,\mathbf{p})} \frac{\Delta_{\mu\nu}(p_0,\mathbf{p})}{(P+Q)^2 P^2}\,, \tag{3.2}$$

$$\Delta_{\mu\nu}(p_0,\mathbf{p}) \equiv \frac{1}{4} \operatorname{Tr}\left[\,\gamma_\mu (\not{P} + \not{Q}) \gamma_\nu \not{P} \gamma_5\right], \tag{3.3}$$

where $\SumInt$ denotes the usual Matsubara sum-integral (bosonic for Q, R and fermionic for P). For right-handed fermions the overall sign is opposite. The Euclidean γ-matrices here are Hermitean, with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and we have defined $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$. It is perhaps appropriate to mention that it is possible to have an "anomalous" term with two gauge field legs only, since the chemical potential effectively acts as a third leg.

To evaluate the sum over ω_n in eq. (3.2), we note that there are only single poles in p_0 and that we can thus use the contour formula

$$T \sum_{n \text{ odd}} f(n\pi T + i\mu) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi} f(p_0) + \sum_{\operatorname{Im} z < 0} \frac{i \operatorname{Res}[f(z)]}{e^{i\beta z + \beta\mu} + 1} - \sum_{\operatorname{Im} z > 0} \frac{i \operatorname{Res}[f(z)]}{e^{-i\beta z - \beta\mu} + 1}\,, \tag{3.4}$$

where the sums are over the poles z of $f(z)$. The first term is independent of T and μ; we ignore this zero-temperature vacuum part here. The latter two terms are ultraviolet and infrared finite, and require no regularization. Thus we can work in exactly four dimensions (as already implicitly assumed above), whereby $\operatorname{Tr}[\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta] = 4\epsilon_{\mu\nu\alpha\beta}$, with $\epsilon_{0123} = +1$.

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Picking up the four poles; shifting integration variables in three of them[2] as $\mathbf{p} \to -\mathbf{p}$, $\mathbf{p} \to \mathbf{p} - \mathbf{q}$, $\mathbf{p} \to -\mathbf{p} - \mathbf{q}$; and employing the facts that $\Delta_{\mu\nu}(-p_0, -\mathbf{p}) = -\Delta_{\mu\nu}(p_0, \mathbf{p})$, $\Delta_{\mu\nu}(p_0 - q_0, \mathbf{p} - \mathbf{q}) = \Delta_{\mu\nu}(p_0, \mathbf{p})$, $\Delta_{\mu\nu}(-p_0 - q_0, -\mathbf{p} - \mathbf{q}) = -\Delta_{\mu\nu}(p_0, \mathbf{p})$; the thermal part of eq. (3.2) becomes

$$\Gamma_{\mu\nu}(Q) = \int_{\mathbf{p}} \frac{N_-(\omega_{\mathbf{p}})}{4\omega_{\mathbf{p}}^2} \Delta_{\mu\nu}(-i\omega_{\mathbf{p}}, \mathbf{p}) \left(\frac{1}{v^E \cdot Q + \frac{Q^2}{2\omega_{\mathbf{p}}}} - \frac{1}{v^E \cdot Q - \frac{Q^2}{2\omega_{\mathbf{p}}}} \right), \tag{3.5}$$

where we have denoted $\int_{\mathbf{p}} \equiv \int \mathrm{d}^3 p/(2\pi)^3$, $N_-(\omega_{\mathbf{p}}) \equiv n_{\mathrm{F}}(\omega_{\mathbf{p}} - \mu) - n_{\mathrm{F}}(\omega_{\mathbf{p}} + \mu)$, $n_{\mathrm{F}}(\omega_{\mathbf{p}}) \equiv 1/(e^{\beta\omega_{\mathbf{p}}} + 1)$, $v_\mu^E \equiv (-i, p_i/\omega_{\mathbf{p}})$, $\omega_{\mathbf{p}} \equiv |\mathbf{p}|$, and $v^E \cdot Q \equiv v_\mu^E Q_\mu$. Note that no approximations have been made so far for the thermal part.

The next step is to carry out a small coupling expansion. In other words, we look for the leading term in the expansion in small $|Q|/\omega_{\mathbf{p}}$, where parametrically (after the analytic continuation to follow presently) Q is a soft scale, $|Q| \lesssim \max(gT, g\mu)$, while the integration variable gets its contributions from the hard scales, $\omega_{\mathbf{p}} \sim \max(T, \mu)$. The leading term in the expansion of the denominators in eq. (3.5) obviously cancels, but the next-to-leading term is non-vanishing, and multiplied by $\Delta_{\mu\nu}(-i\omega_{\mathbf{p}}, \mathbf{p}) = \omega_{\mathbf{p}} \epsilon_{\mu\alpha\nu\beta} Q_\alpha v_\beta^E$. Thus, we obtain

$$\Gamma_{\mu\nu}(Q) \approx \epsilon_{\mu\nu\alpha\beta}\, Q^2 \int_{\mathbf{p}} \frac{N_-(\omega_{\mathbf{p}})}{4\omega_{\mathbf{p}}^2} \frac{Q_\alpha v_\beta^E}{(v^E \cdot Q)^2}\,. \tag{3.6}$$

At this point we carry out analytic continuation to Minkowski spacetime. Any Euclidean Lorentz-vector can be written as $f_\alpha^E = \Lambda_{\alpha\beta} f^{M\beta}$, with $\Lambda_{\alpha\beta} = \mathrm{diag}(-i, -1, -1, -1)$. Furthermore, as mentioned in eq. (2.11), there is an overall minus-sign between $\mathcal{L}_E$ and $\mathcal{L}_M$. We also introduce an angular integration $\int_v \equiv \int \mathrm{d}\Omega_v / 4\pi$, where $v^\mu = (1, v^i)$, $v_\mu v^\mu = 0$, and the integral is over the directions Ω_v of v^i, and note that the radial integration in eq. (3.6) can be carried out exactly, with the result

$$\int_{\mathbf{p}} \frac{N_-(\omega_{\mathbf{p}})}{\omega_{\mathbf{p}}^2} = -\int_{\mathbf{p}} \frac{N_-'(\omega_{\mathbf{p}})}{\omega_{\mathbf{p}}} = \frac{\mu}{2\pi^2}\,. \tag{3.7}$$

Summing over all fermions, adding the known non-anomalous bosonic terms [14], and going furthermore to x-space, we arrive at our final action:

$$S_M = \int_x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] + S_{\mathrm{HTL}}\,, \tag{3.8}$$

$$S_{\mathrm{HTL}} = \int_{x,v} \left[-j_E' v^\mu B_\mu - \frac{m_E'^2}{4} F_{\alpha\mu} \frac{v^\alpha v^\beta}{(v \cdot \partial)^2} F_\beta{}^\mu + c_E' \frac{g'^2}{32\pi^2} \widetilde{F}_{\alpha\mu}(x) \frac{v^\alpha}{(v \cdot \partial)^2} \partial^2 B^\mu(x) \right], \tag{3.9}$$

where $\widetilde{F}_{\alpha\mu} \equiv \epsilon_{\alpha\mu\beta\nu} F^{\beta\nu}$ and $\partial^2 \equiv \partial \cdot \partial \equiv \partial^\mu \partial_\mu$. The coefficients $m_E'^2, j_E', c_E'$ are the same as in eqs. (2.6), (2.7), and (2.9).[3] The integrals $\int_v (\ldots)$ are collected in the appendix.

[2] Shifting integration variables is safe provided that each individual term is finite, as is the case here. We have however checked the outcome also by not carrying out any shifts but just expanding in $Q/\omega_{\mathbf{p}}$.

[3] We note that a single chiral fermion contributes to the parameters as $j_E' = g'\mu Y (T^2 + \mu^2/\pi^2)/12$, $m_E'^2 = g'^2 Y^2 (T^2/3 + \mu^2/\pi^2)/8$, $c_E' = -\mu Y^2 H/2$, where $H = +1$ (-1) for right-handed (left-handed) fermions.

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It is the last term in eq. (3.9) which is the new one. For more symmetry, one could replace $B^\mu \to v^\beta/(v\cdot\partial)F_\beta{}^\mu$ in it, since the extra term $-[\partial^\mu/(v\cdot\partial)]v\cdot B$ thus introduced vanishes after partial integration and neglectance of surface terms.

Note that eq. (3.9) is gauge invariant in time-dependent gauge transformations, unlike eq. (2.1). Nevertheless, by employing the identities in eqs. (A.11), (A.12), it is straightforward to rewrite S_M in a form where it is obvious that the hypercharge part of eq. (2.1) is recovered in the static limit, up to corrections of order $\mathcal{O}(|\partial_0|/|\nabla|)$.

For completeness, we remark that although we have shown in eq. (3.9) only the terms with the three largest coefficients, $j'_E \sim \mathcal{O}(\mu T^2), m'^2_E \sim \mathcal{O}(T^2)$, and $c'_E \sim \mathcal{O}(\mu)$, some additional operators are known as well: there is a non-anomalous but charge-conjugation violating "cubic" operator with a coefficient $\mathcal{O}(\mu)$ [25], and it appears that even "quartic" operators with a coefficient $\mathcal{O}(1)$ could be added, by reformulating the HTL action in terms of a certain matrix-valued kinetic theory [26].

Since the action in eqs. (3.8), (3.9) is quadratic in the fields, its contents can equivalently be expressed through equations of motion. It is useful to express the equations of motion in a form following from the identities in eqs. (A.11), (A.12). We obtain

$$\partial^\mu F_{\mu\nu}(x) = \int_v \left[j'_E v_\nu - m'^2_E \frac{v_\nu v^\alpha}{v\cdot\partial} F_{\alpha 0}(x) - c'_E \epsilon_{0\nu\alpha\beta} \frac{g'^2}{8\pi^2} \frac{v^\alpha}{v\cdot\partial} \partial^2 B^\beta(x) \right]. \tag{3.10}$$

The integrals have the properties $\int_v v^\alpha = \delta^\alpha{}_0$ and $(1-\delta^\alpha{}_0)\int_v v^\alpha/v\cdot\partial \propto \partial^\alpha$ (cf. the appendix), which imply that the right-hand side is divergenceless (or transverse) with respect to ∂^ν.

We should like to stress already at this point that there are circumstances where higher order corrections to eq. (3.10) can become important. Let us recall, to start with, that standard HTL structures like the second term in eq. (3.10) can be reproduced [27] by classical kinetic theory, or "Vlasov equations", of the type

$$\left[p\cdot\partial_x - \frac{g'Y}{2} p^\alpha F_{\alpha\beta} \partial_p^\beta \right] f(x,p) \equiv 0\,, \tag{3.11}$$

$$\partial^\mu F_{\mu\nu} \equiv \frac{g'Y}{2} \int_{\mathbf{p}} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_\nu\, f(x,p)\,, \tag{3.12}$$

by solving them perturbatively to second order in g' with the "initial condition" that for $g' \to 0$ the solution reads $f(x,p) \equiv f_{(0)}(x,p) \equiv 4\pi\delta(p^2) n_{\mathrm{F}}(|p_0| - \mathrm{sign}(p_0)\mu)$. One may then expect higher order interactions to generate a Boltzmann type collision term on the right-hand side of eq. (3.11). Indeed, effective theories which such a structure have recently been analysed in Ref. [28], and the collision term does turn out to be important in many contexts. We will return to this issue presently.

Finally, the question arises whether the last term in eq. (3.10) can also be expressed through Vlasov equations. In principle this indeed is the case: for instance, we can "by brute force" generalise $f(x,p)$ to a two-index Lorentz tensor, and the Vlasov equations to

$$\left[p\cdot\partial_x - \frac{g'Y}{2} p^\alpha F_{\alpha\beta} \partial_p^\beta \right] f^{\gamma\delta}(x,p) - \frac{g'Y}{2} \epsilon^{\gamma\delta}{}_{\alpha\rho} \partial_x^2 B^\alpha \partial_\sigma^p f^{\rho\sigma}(x,p) \equiv 0\,, \tag{3.13}$$

$$\partial^\mu F_{\mu\nu} \equiv \frac{g'Y}{2} \int_{\mathbf{p}} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi} \Big[\eta_{\gamma\delta}\, p_\nu + H(\eta_{\gamma\nu}\, p_\delta - \eta_{\delta\nu}\, p_\gamma) \Big] f^{\gamma\delta}(x,p)\,, \tag{3.14}$$

$$f^{\gamma\delta}_{(0)}(x,p) \equiv \eta^{\gamma\delta} \pi \delta(p^2) n_{\mathrm{F}}(|p_0| - \mathrm{sign}(p_0)\mu)\,, \tag{3.15}$$

where $H = +1$ (-1) for right-handed (left-handed) fermions. Solving this set perturbatively to second order in g', it is easy to check that eq. (3.10) is reproduced (with a single chiral fermion contributing as specified in footnote 3). Nevertheless the new parts in these equations are not particularly satisfactory: eq. (3.13) generates a distribution which contains an ugly $\delta'(p^2)$ and is not gauge invariant. While both problems disappear after the construction of the current in eq. (3.14), it should be possible to find a more compelling formulation which also has a physically understandable interpretation.

4. On the growth rate of instabilities

In order to illustrate the effects that may originate from the non-local HTL structures, we inspect the spatial part of eq. (3.10) (i.e. $\nu = 1,2,3$). We employ a class of gauges where B_0 is constant, so that $F_{k0} = -\partial_0 B_k$, where k is a spatial index.

We start by considering the static limit. Then eq. (3.10) becomes ($k = 1,2,3$)

$$[\nabla^2 \delta_{ki} - \partial_k \partial_i] B_i(x) = c'_E \frac{g'^2}{16\pi^2} \epsilon_{kij} F_{ij}(x)\,. \tag{4.1}$$

Going into momentum space [$B_k(x) = \int_{\mathbf{q}} \tilde{B}_k(\mathbf{q}) \exp(-i\mathbf{q}\cdot\mathbf{x})$], and choosing a frame where $\mathbf{q} = (0,0,q^3)$, we obtain equations for the transverse components $\tilde{B}_1, \tilde{B}_2$. They have a non-trivial solution provided that

$$|\mathbf{q}| = \pm |q_{\mathrm{CS}}|\,, \qquad q_{\mathrm{CS}} \equiv \frac{g'^2 c'_E}{8\pi^2}\,. \tag{4.2}$$

Going back to configuration space, the solution reads

$$B_1(x^3) = C\, \cos\left[q_{\mathrm{CS}}(x^3 - x^3_0)\right], \qquad B_2(x^3) = C\, \sin\left[q_{\mathrm{CS}}(x^3 - x^3_0)\right], \tag{4.3}$$

where C, x^3_0 are constants. This is nothing but the so-called (static) Chern-Simons wave [29].

Consider then the dynamical situation. We transform eq. (3.10) to Fourier space with respect to space coordinates but keep the time coordinate in configuration space. Building on eq. (4.1), we look for transverse modes which satisfy the eigenvalue equations ($k' = 1,2$)

$$\mathbf{q}^2 \tilde{B}_{k'} + i q_{\mathrm{CS}} \epsilon_{k'ij} q_i \tilde{B}_j = \lambda \tilde{B}_{k'}\,. \tag{4.4}$$

Two non-trivial solutions exist, with the eigenvalues

$$\lambda = \mathbf{q}^2 \pm q_{\mathrm{CS}} |\mathbf{q}|\,. \tag{4.5}$$

Thus, for long wavelengths, $|\mathbf{q}| < |q_{\mathrm{CS}}|$, there exists an unstable branch with $\lambda < 0$.

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How fast do the modes with $\lambda < 0$ grow? To find out, we assume that the time evolution is very slow, $|\partial_t^2| \ll |\lambda|$, and justify this assumption a posteriori. In this situation, we can approximate the complicated HTL structures through the leading terms in time derivatives, and the problem becomes tractable.

Inspecting the integrals in the appendix, it can be seen that both the 2nd and the 3rd term on the right-hand side of eq. (3.10) have a term linear in time derivatives. For a solution of eq. (4.4), however, the third term gives a contribution suppressed by $\mathcal{O}[(\mathbf{q}^2 - \lambda)/m_E'^2] \sim \mathcal{O}[q_{\rm CS}^2/m_E'^2]$ with respect to the second term. This is very small for weak coupling, $\sim g'^2\mu^2/\max(T^2,\mu^2)\pi^4$, and can safely be ignored. Thus the effect comes from the 2nd term, and we obtain

$$-\lambda \tilde{B}_{k'} \approx \frac{\pi}{4|\mathbf{q}|} m_E'^2 \partial_t \tilde{B}_{k'} \,. \tag{4.6}$$

To summarise, modes with $\lambda < 0$ grow exponentially, $\tilde{B}_{k'} \sim \exp(\Gamma t)$, with the rate

$$\Gamma \approx \frac{4|\lambda \mathbf{q}|}{\pi m_E'^2} \sim \frac{q_{\rm CS}^2}{m_E'^2}|\mathbf{q}| \,. \tag{4.7}$$

Given that $q_{\rm CS}^2/m_E'^2 \ll 1$, the assumption of slow growth ($\Gamma^2 \ll \mathbf{q}^2$) is indeed justified.

It is important to realise at this point, though, that higher order corrections may give large contributions on the right-hand side of eq. (4.6), as already mentioned above. In fact, it could happen that summing an infinite set of higher order loop contributions effectively shields the scale $|\mathbf{q}|$ in the denominator of $\pi m_E'^2/4|\mathbf{q}|$ by a constant $\sim g'^4 \ln(1/g')T$, whereby the right-hand side goes over into $\sigma' \partial_t \tilde{B}_{k'}$, where σ' is the hyperelectric conductivity, $\sigma' \sim T/g'^2 \ln(1/g')$ [30]. Another way to think of the issue[4] is that interactions may generate a "thermal width" $\Gamma_{\rm th}$ of order $\Gamma_{\rm th} \sim g'^4 \ln(1/g')T$ for the hard on-shell particles [31] and that, if $q_0 \to q_0 + i\Gamma_{\rm th}$ and $|\mathbf{q}| \ll \Gamma_{\rm th}$ in eq. (A.10), then the static limit would become $-i/\Gamma_{\rm th}$ rather than $-i\pi/2|\mathbf{q}|$, amounting to the same shielding. Since $|\mathbf{q}| \sim q_{\rm CS} \sim g'^2\mu$ and $\Gamma_{\rm th} \sim g'^4 \ln(1/g')T$, we should formally assume $|\mathbf{q}| \gg \Gamma_{\rm th}$ so that the shielding is irrelevant, but since typically $\mu \ll g'^2 T$ in cosmology, the formal hierarchy may get reversed so that we indeed find ourselves in the situation $|\mathbf{q}| \ll \Gamma_{\rm th}$. In fact, inserting numerical values relevant for the Standard Model, it appears that having $|\sum_{i=1}^{n_{\rm G}}(\mu_{L_i} - \mu_{R_i})|/T \lesssim 0.5$ already brings us to the reversed situation.

The estimates for the growth rate of hypermagnetic fields that were presented in refs. [6, 11] were based on the contribution of the hyperelectric conductivity σ', rather than eq. (4.6), and should thus be correct under the phenomenologically relevant circumstances $\mu \ll T$. (The expansion of the Universe as well as the time-dependence of the chemical potentials, due to the backreaction via the hypermagnetic part of the anomaly equation as well as the chirality flipping processes induced by the electron Yukawa coupling, were also taken into account.) We note that the corresponding growth rate is much *larger* than eq. (4.7), since $|\mathbf{q}|$ gets replaced with $2\Gamma_{\rm th}/\pi \gg |\mathbf{q}|$.

[4] We stress that the discussion here is only qualitative in nature, and omits important points.

5. Conclusions

We have addressed here the coupling between hypermagnetic fields and fermionic chemical potentials in the standard electroweak theory at high temperatures. This problem has phenomenological relevance in cosmology, provided that a lepton asymmetry and/or primordial hypermagnetic fields exist at temperatures above the electroweak phase transition, but is also related to some intriguing theoretical issues, such as that the coupling discussed seems to allow for a sharp distinction between the high-temperature and low-temperature phases of the electroweak theory [32].

Concretely, we have generalised the standard abelian Chern-Simons term to an apparently Lorentz invariant form, which can be added to the Hard Thermal Loop action describing the real-time dynamics of the hypermagnetic fields (eq. (3.9)). We have also analysed the unstable exponentially growing solutions that the resulting equations of motion have. Our conclusion is that for such solutions, the deviation of the anomalous term from its standard static form is in fact insignificant in practice (cf. the discussion preceding eq. (4.6)), so that the ignoring of this deviation in previous studies appears well justified in retrospective.

The problem with the Hard Thermal Loop equations of motion is that higher order corrections to their non-anomalous part turn out to be very important for the small values of chemical potentials that are assumed to appear in cosmology, $\mu \ll T$. (The wave vectors of the growing modes are proportional to differences of chemical potentials and thus very much smaller than the temperature in this situation.) In particular, the fact that a finite conductivity is expected to be generated through summing infinitely many high order loop corrections modifies the growth rate of the unstable solutions significantly. (The existence of unstable modes is not affected.) Nevertheless, it seems to us that the conclusion mentioned above, namely that it is safe to use the static limit of the Chern-Simons term under phenomenologically relevant circumstances, continues to be valid. To be sure, it would of course be interesting to develop a numerical framework where both the Hard Thermal Loop effects discussed in this paper, and higher order corrections such as conductivity, can be incorporated simultaneously, to allow for a more precise study of growing hypermagnetic fields.

Finally, we remark that the growing hypermagnetic fields have a form which generates a non-zero value for the hypermagnetic topological charge density, $\sim g'^2 F_{\mu\nu}\widetilde{F}_{\mu\nu}$. This means that fermion number densities and, consequently, chemical potentials, evolve as dictated by the anomaly equation, in a way which stops the growth at some point. These processes have been analysed in refs. [6, 11].

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A. Basic integrals

Although well-known [24, 13, 14], we collect here the basic velocity integrals needed in this paper. Assuming implicitly that the frequency is replaced everywhere through $q_0 \to$

$q_0 + i0^+$, as is relevant for retarded Green's functions, the integrals read $(i, j = 1, 2, 3)$

$$\int_v = 1\,, \tag{A.1}$$

$$\int_v v^i = 0\,, \tag{A.2}$$

$$\int_v v^i v^j = \frac{1}{3}\delta^{ij}\,, \tag{A.3}$$

$$\int_v \frac{1}{v\cdot q} = L(q)\,, \tag{A.4}$$

$$\int_v \frac{v^i}{v\cdot q} = \frac{q^i}{|\mathbf{q}|^2}\Big[-1 + q_0 L(q)\Big]\,, \tag{A.5}$$

$$\int_v \frac{v^i v^j}{v\cdot q} = \frac{L(q)}{2}\left(\delta^{ij} - \frac{q^i q^j}{|\mathbf{q}|^2}\right) + \frac{q_0}{2|\mathbf{q}|^2}\Big[1 - q_0 L(q)\Big]\left(\delta^{ij} - 3\frac{q^i q^j}{|\mathbf{q}|^2}\right), \tag{A.6}$$

$$\int_v \frac{1}{(v\cdot q)^2} = \frac{1}{q^2}\,, \tag{A.7}$$

$$\int_v \frac{v^i}{(v\cdot q)^2} = \frac{q^i}{|\mathbf{q}|^2}\Big[\frac{q_0}{q^2} - L(q)\Big]\,, \tag{A.8}$$

$$\int_v \frac{v^i v^j}{(v\cdot q)^2} = \frac{1}{2q^2}\left(\delta^{ij} - \frac{q^i q^j}{|\mathbf{q}|^2}\right) - \frac{1}{2|\mathbf{q}|^2}\Big[1 - 2q_0 L(q) + \frac{q_0^2}{q^2}\Big]\left(\delta^{ij} - 3\frac{q^i q^j}{|\mathbf{q}|^2}\right), \tag{A.9}$$

where $v^\mu \equiv (1, v^i)$, $q \equiv (q^0, \mathbf{q})$, our metric convention is $(+---)$, and

$$L(q) \equiv \frac{1}{2|\mathbf{q}|}\ln\frac{q_0 + |\mathbf{q}|}{q_0 - |\mathbf{q}|} \approx -\frac{i\pi}{2|\mathbf{q}|} + \frac{q_0}{|\mathbf{q}|^2} + \frac{q_0^3}{3|\mathbf{q}|^4} + \cdots\,. \tag{A.10}$$

Integrals with higher powers of $v\cdot q$ in the denominator can be obtained through the partial derivatives $\partial/\partial q_0$. The following identities (which can be derived by certain partial integrations, or by explicit inspection) are often very useful:

$$\int_v \frac{v^\alpha q^\beta}{(v\cdot q)^2}\epsilon_{\alpha\beta\mu\nu} I^\mu J^\nu = \int_v \frac{v^\alpha}{v\cdot q}\epsilon_{0\alpha\mu\nu} I^\mu J^\nu\,, \tag{A.11}$$

$$\int_v \frac{v^\alpha v^\beta}{(v\cdot q)^2} q_{[\alpha} I_{\mu]} q_{[\beta} J_{\nu]}\eta^{\mu\nu} = 2\int_v \frac{v^\alpha v^\beta}{v\cdot q} I_\alpha q_{[\beta} J_{0]} = 2\int_v \frac{v^\alpha v^\beta}{v\cdot q} q_{[\alpha} I_{0]} J_\beta\,. \tag{A.12}$$

Here $q_{[\alpha} I_{\mu]} \equiv q_\alpha I_\mu - q_\mu I_\alpha$, and I, J are arbitrary Lorentz vectors.

References

[1] A.N. Redlich and L.C.R. Wijewardhana, *Induced Chern-Simons terms at high temperatures and finite densities*, *Phys. Rev. Lett.* **54** (1985) 970;
K. Tsokos, *Topological mass terms and the high temperature limit of chiral gauge theories*, *Phys. Lett.* **B 157** (1985) 413.

[2] A.J. Niemi and G.W. Semenoff, *A comment on 'Induced Chern-Simons terms at high temperatures and finite densities'*, *Phys. Rev. Lett.* **54** (1985) 2166.

[3] S. Deser, R. Jackiw and S. Templeton, *Topologically massive gauge theories*, *Ann. Phys. (NY)* **140** (1982) 372.

[4] G. 't Hooft, *Symmetry breaking through Bell-Jackiw anomalies*, *Phys. Rev. Lett.* **37** (1976) 8.

[5] B.A. Campbell, S. Davidson, J.R. Ellis and K.A. Olive, *On the baryon, lepton-flavor and right-handed electron asymmetries of the universe*, *Phys. Lett.* **B 297** (1992) 118 [hep-ph/9302221];
J.M. Cline, K. Kainulainen and K.A. Olive, *On the erasure and regeneration of the primordial baryon asymmetry by sphalerons*, *Phys. Rev. Lett.* **71** (1993) 2372 [hep-ph/9304321].

[6] M. Joyce and M.E. Shaposhnikov, *Primordial magnetic fields, right electrons and the abelian anomaly*, *Phys. Rev. Lett.* **79** (1997) 1193 [astro-ph/9703005].

[7] M. Christensson, M. Hindmarsh and A. Brandenburg, *Scaling laws in decaying helical 3D magnetohydrodynamic turbulence*, *Astron. Nachr.* **326** (2005) 393 [astro-ph/0209119];
L. Campanelli, *Scaling laws in magnetohydrodynamic turbulence*, *Phys. Rev.* **D 70** (2004) 083009 [astro-ph/0407056];
R. Banerjee and K. Jedamzik, *The evolution of cosmic magnetic fields: from the very early universe, to recombination, to the present*, *Phys. Rev.* **D 70** (2004) 123003 [astro-ph/0410032].

[8] P. Elmfors, K. Enqvist and K. Kainulainen, *Strongly first order electroweak phase transition induced by primordial hypermagnetic fields*, *Phys. Lett.* **B 440** (1998) 269 [hep-ph/9806403];
K. Kajantie, M. Laine, J. Peisa, K. Rummukainen and M.E. Shaposhnikov, *The electroweak phase transition in a magnetic field*, *Nucl. Phys.* **B 544** (1999) 357 [hep-lat/9809004].

[9] D. Comelli, D. Grasso, M. Pietroni and A. Riotto, *The sphaleron in a magnetic field and electroweak baryogenesis*, *Phys. Lett.* **B 458** (1999) 304 [hep-ph/9903227].

[10] G. Piccinelli and A. Ayala, *Electroweak baryogenesis and primordial hypermagnetic fields*, *Lect. Notes Phys.* **646** (2004) 293 [hep-ph/0404033];
L. Campanelli, P. Cea, G.L. Fogli and L. Tedesco, *Baryogenesis and CP-violating domain walls in the background of a magnetic field*, astro-ph/0505531.

[11] M. Giovannini and M.E. Shaposhnikov, *Primordial magnetic fields, anomalous isocurvature fluctuations and big bang nucleosynthesis*, *Phys. Rev. Lett.* **80** (1998) 22 [hep-ph/9708303]; *Primordial hypermagnetic fields and the triangle anomaly*, *Phys. Rev.* **D 57** (1998) 2186 [hep-ph/9710234].

[12] J.B. Rehm and K. Jedamzik, *Big bang nucleosynthesis with matter/antimatter domains*, *Phys. Rev. Lett.* **81** (1998) 3307 [astro-ph/9802255]; *Limits on cosmic matter-antimatter domains from big bang nucleosynthesis*, *Phys. Rev.* **D 63** (2001) 043509 [astro-ph/0006381];
H. Kurki-Suonio and E. Sihvola, *Constraining antimatter domains in the early universe with big bang nucleosynthesis*, *Phys. Rev. Lett.* **84** (2000) 3756 [astro-ph/9912473]; *Antimatter regions in the early universe and big bang nucleosynthesis*, *Phys. Rev.* **D 62** (2000) 103508 [astro-ph/0006448].

[13] R.D. Pisarski, *Scattering amplitudes in hot gauge theories*, *Phys. Rev. Lett.* **63** (1989) 1129;
J. Frenkel and J.C. Taylor, *High temperature limit of thermal QCD*, *Nucl. Phys.* **B 334** (1990) 199;
E. Braaten and R.D. Pisarski, *Soft amplitudes in hot gauge theories: a general analysis*, *Nucl. Phys.* **B 337** (1990) 569;
J.C. Taylor and S.M.H. Wong, *The effective action of hard thermal loops in QCD*, *Nucl. Phys.* **B 346** (1990) 115.

[14] J. Frenkel and J.C. Taylor, *Hard thermal QCD, forward scattering and effective actions*, *Nucl. Phys.* **B 374** (1992) 156;
E. Braaten and R.D. Pisarski, *Simple effective lagrangian for hard thermal loops*, *Phys. Rev.* **D 45** (1992) 1827.

[15] P.H. Ginsparg, *First order and second order phase transitions in gauge theories at finite temperature*, *Nucl. Phys.* **B 170** (1980) 388;
T. Appelquist and R.D. Pisarski, *High-temperature Yang-Mills theories and three-dimensional quantum chromodynamics*, *Phys. Rev.* **D 23** (1981) 2305.

[16] K. Kajantie, M. Laine, K. Rummukainen and M.E. Shaposhnikov, *High temperature dimensional reduction and parity violation*, *Phys. Lett.* **B 423** (1998) 137 [hep-ph/9710538].

[17] A. Gynther, *The electroweak phase diagram at finite lepton number density*, *Phys. Rev.* **D 68** (2003) 016001 [hep-ph/0303019].

[18] P. Arnold, D. Son and L.G. Yaffe, *The hot baryon violation rate is $O(\alpha_w^5 T^4)$*, *Phys. Rev.* **D 55** (1997) 6264 [hep-ph/9609481].

[19] D. Bödeker, *On the effective dynamics of soft non-abelian gauge fields at finite temperature*, *Phys. Lett.* **B 426** (1998) 351 [hep-ph/9801430]; *From hard thermal loops to Langevin dynamics*, *Nucl. Phys.* **B 559** (1999) 502 [hep-ph/9905239].

[20] D. Bödeker, G.D. Moore and K. Rummukainen, *Chern-Simons number diffusion and hard thermal loops on the lattice*, *Phys. Rev.* **D 61** (2000) 056003 [hep-ph/9907545].

[21] P. Arnold and L.G. Yaffe, *Non-perturbative dynamics of hot non-abelian gauge fields: beyond leading log approximation*, *Phys. Rev.* **D 62** (2000) 125013 [hep-ph/9912305];
D. Bödeker, *Perturbative and non-perturbative aspects of the non-abelian Boltzmann-Langevin equation*, *Nucl. Phys.* **B 647** (2002) 512 [hep-ph/0205202].

[22] G.D. Moore, *Sphaleron rate in the symmetric electroweak phase*, *Phys. Rev.* **D 62** (2000) 085011 [hep-ph/0001216]; *Do we understand the sphaleron rate?*, hep-ph/0009161.

[23] V.A. Kuzmin, V.A. Rubakov and M.E. Shaposhnikov, *On the anomalous electroweak baryon number nonconservation in the early universe*, *Phys. Lett.* **B 155** (1985) 36.

[24] V.P. Silin, *On the electromagnetic properties of a relativistic plasma*, *Sov. Phys. JETP* **11** (1960) 1136;
V.V. Klimov, *Collective excitations in a hot quark gluon plasma*, *Sov. Phys. JETP* **55** (1982) 199;
H.A. Weldon, *Covariant calculations at finite temperature: the relativistic plasma*, *Phys. Rev.* **D 26** (1982) 1394.

[25] D. Bödeker and M. Laine, *Finite baryon density effects on gauge field dynamics*, *JHEP* **09** (2001) 029 [hep-ph/0108034].

[26] M. Laine and C. Manuel, *A remark on non-abelian classical kinetic theory*, *Phys. Rev.* **D 65** (2002) 077902 [hep-ph/0111113];
C. Manuel and S. Mrowczynski, *QCD effective actions from the solutions of the transport equations*, *Phys. Rev.* **D 67** (2003) 014015 [hep-ph/0206209].

[27] J.P. Blaizot and E. Iancu, *Kinetic equations for long wavelength excitations of the quark — gluon plasma*, *Phys. Rev. Lett.* **70** (1993) 3376 [hep-ph/9301236];
P.F. Kelly, Q. Liu, C. Lucchesi and C. Manuel, *Deriving the hard thermal loops of QCD from classical transport theory*, *Phys. Rev. Lett.* **72** (1994) 3461 [hep-ph/9403403];

JHEP10(2005)056

F.T. Brandt, J. Frenkel and J.C. Taylor, *High temperature QCD and the classical Boltzmann equation in curved space-time*, *Nucl. Phys.* **B 437** (1995) 433 [hep-th/9411130].

[28] P. Arnold, G.D. Moore and L.G. Yaffe, *Effective kinetic theory for high temperature gauge theories*, *JHEP* **01** (2003) 030 [hep-ph/0209353].

[29] V.A. Rubakov and A.N. Tavkhelidze, *Stable anomalous states of superdense matter in gauge theories*, *Phys. Lett.* **B 165** (1985) 109;
V.A. Rubakov, *On the electroweak theory at high fermion density*, *Prog. Theor. Phys.* **75** (1986) 366.

[30] P. Arnold, G.D. Moore and L.G. Yaffe, *Transport coefficients in high temperature gauge theories, I. Leading-log results*, *JHEP* **11** (2000) 001 [hep-ph/0010177]; *Transport coefficients in high temperature gauge theories, II. Beyond leading log*, *JHEP* **05** (2003) 051 [hep-ph/0302165].

[31] M.A. Valle Basagoiti, *Transport coefficients and ladder summation in hot gauge theories*, *Phys. Rev.* **D 66** (2002) 045005 [hep-ph/0204334];
G. Aarts and J.M. Martínez Resco, *Ward identity and electrical conductivity in hot QED*, *JHEP* **11** (2002) 022 [hep-ph/0209048].

[32] M. Laine and M.E. Shaposhnikov, *An all-order discontinuity at the electroweak phase transition*, *Phys. Lett.* **B 463** (1999) 280 [hep-th/9907194].

JHEP10(2005)056